

# A Taylor-like Expansion of a Commutator with a Function of Self-adjoint, Pairwise Commuting Operators

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## Abstract

Let  $A$  be a  $\nu$ -vector of self-adjoint, pairwise commuting operators and  $B$  a bounded operator of class  $C^{n_0}(A)$ . We prove a Taylor-like expansion of the commutator  $[B, f(A)]$  for a large class of functions  $f: \mathbb{R}^\nu \rightarrow \mathbb{R}$ , generalising the one-dimensional result where  $A$  is just a self-adjoint operator. This is done using almost analytic extensions and the higher-dimensional Helffer-Sjöstrand formula.

*Keywords:* Commutator expansions, functional calculus, almost analytic extensions, Helffer-Sjöstrand formula.

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# 1 Introduction

It is well-known that if  $A$  is a self-adjoint operator,  $B$  is a bounded operator of class  $C^{n_0}(A)$  in the sense of [1] and  $f$  satisfies  $|f^{(n)}(x)| \leq C_n \langle x \rangle^{s-n}$  for all  $n$ , then for  $0 \leq t_1 \leq n_0$ ,  $0 \leq t_2 \leq 1$  with  $s + t_1 + t_2 < n_0$ ,

$$[B, f(A)] = \sum_{k=1}^{n_0-1} \frac{1}{k!} f^{(k)}(A) \operatorname{ad}_A^k(B) + R_{n_0}(A, B)$$

where  $\operatorname{ad}_A^k(B)$  is the  $k$ 'th iterated commutator,  $R_{n_0}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}; \mathcal{H}_A^{t_1})$  and  $\mathcal{H}_A^t$  is defined as  $\mathcal{D}(\langle A \rangle^t)$  equipped with the graph-norm  $\|v\|_t = \|\langle A \rangle^t v\|$  for  $t \geq 0$  and  $\mathcal{H}_A^{-t}$  is the dual space of  $\mathcal{H}_A^t$ . This follows relatively easily from using the (one-dimensional) Helffer-Sjöstrand formula

$$f(A) = \frac{1}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-1} dz, \quad (1)$$

where  $\bar{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$  and  $\tilde{f}$  is an almost analytic extension of  $f$ , and the identity

$$\begin{aligned} [B, f(A)] &= \sum_{k=1}^{n_0-1} \frac{1}{k!} \frac{k!}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (-1)^k (A - z)^{-k-1} dz \\ &\quad + \frac{(-1)^{n_0}}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (A - z)^{-n_0} \operatorname{ad}_A^{n_0}(B) (A - z)^{-1} dz \end{aligned}$$

when  $\frac{k!}{\pi} \int_{\mathbb{C}} \bar{\partial} \tilde{f}(z) (-1)^k (A - z)^{-k-1} dz$  is recognised as  $f^{(k)}(A)$  using (1). Such commutator expansions were first proved in [7]. See e.g. [4] for details. Due to the higher complexity of the general Helffer-Sjöstrand formula, these calculations do not lead directly to the generalised result where  $A$  is a vector of self-adjoint, pairwise commuting operators. However, we will follow the same idea.

The theorem may be viewed as an abstract analogue of pseudo-differential calculus. The one-dimensional version is an often used result, see e.g. [2] and [4]. Apart from the obvious interest in generalising the result to higher dimensions, our improvement has proven useful in the treatment of models in quantum field theory, see [6]. In particular, a lemma in [6] whose proof depends on our result, extends the results of [5] to a larger class of models.

## 2 The setting and result

In the following,  $A = (A_1, \dots, A_\nu)$  is a vector of self-adjoint, pairwise commuting operators acting on a Hilbert space  $\mathcal{H}$ , and  $B \in \mathcal{B}(\mathcal{H})$  is a bounded operator on  $\mathcal{H}$ . We shall use the notion of  $B$  being of class  $C^{n_0}(A)$  introduced in [1]. For notational convenience, we adopt the following convention: If  $0 \leq j \leq \nu$ , then  $\delta_j$  denotes the multi-index  $(0, \dots, 0, 1, 0, \dots, 0)$ , where the 1 is in the  $j$ 'th entry.

**Definition 1.** Let  $n_0 \in \mathbb{N} \cup \{\infty\}$ . Assume that the multi-commutator form defined iteratively by  $\operatorname{ad}_A^0(B) = B$  and  $\operatorname{ad}_A^\alpha(B) = [\operatorname{ad}_A^{\alpha-\delta_j}(B), A_j]$  as a form on  $\mathcal{D}(A_j)$ , where

$\alpha \geq \delta_j$  is a multi-index and  $1 \leq j \leq \nu$ , can be represented by a bounded operator also denoted by  $\text{ad}_A^\alpha(B)$ , for all multi-indices  $\alpha$ ,  $|\alpha| < n_0 + 1$ . Then  $B$  is said to be of class  $C^{n_0}(A)$  and we write  $B \in C^{n_0}(A)$ .

**Remark 2.** The definition of  $\text{ad}_A^\alpha(B)$  does not depend on the order of the iteration since the  $A_j$  are pairwise commuting. We call  $|\alpha|$  the *degree* of  $\text{ad}_A^\alpha(B)$ .

In the following,  $\mathcal{H}_A^s := D(|H|^s)$  for  $s \geq 0$  will be used to denote the scale of spaces associated to  $A$ . For negative  $s$ , we define  $\mathcal{H}_A^s := (\mathcal{H}_A^{-s})^*$ .

**Theorem 3.** Assume that  $B \in C^{n_0}(A)$  for some  $n_0 \geq n + 1 \geq 1$ ,  $0 \leq t_1 \leq n + 1$ ,  $0 \leq t_2 \leq 1$  and that  $\{f_\lambda\}_{\lambda \in I}$  satisfies

$$\forall \alpha \exists C_\alpha: |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}$$

uniformly in  $\lambda$  for some  $s \in \mathbb{R}$  such that  $t_1 + t_2 + s < n + 1$ . Then

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \text{ad}_A^\alpha(B) + R_{\lambda,n}(A, B)$$

as an identity on  $\mathcal{D}(\langle A \rangle^s)$ , where  $R_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})$  and there exist a constant  $C$  independent of  $A$ ,  $B$  and  $\lambda$  such that

$$\|R_{\lambda,n}(A, B)\|_{\mathcal{B}(\mathcal{H}_A^{-t_2}, \mathcal{H}_A^{t_1})} \leq C \sum_{|\alpha|=n+1} \|\text{ad}_A^\alpha(B)\|.$$

**Remark 4.** A similar statement holds with the  $\text{ad}_A^\alpha(B)$  and  $\partial^\alpha f_\lambda(A)$  interchanged at the cost of a sign correction given by  $(-1)^{|\alpha|-1}$ , and the corresponding remainder term  $R'_{\lambda,n}(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_1}, \mathcal{H}_A^{t_2})$ . This can be seen either by proving it analogously or by taking the adjoint equation and replacing  $B$  by  $-B$ .

**Remark 5.** If  $k \leq t_1$  and  $n_0 \geq n + 1 + k$ , then  $R_{\lambda,n}(A, B)$  can be replaced by  $R_{\lambda,n}^k(A, B) \in \mathcal{B}(\mathcal{H}_A^{-t_2+k}, \mathcal{H}_A^{t_1-k})$ . This can be seen by commuting  $|A - z|^{-2}$  and  $\text{ad}_A^\alpha(B)$  in the terms of the remainder, see page 8.

### 3 The Proof

Let  $z \in \mathbb{C}^\nu$ ,  $\text{Im } z \neq 0$ ,  $1 \leq \ell \leq \nu$  and  $g, g_\ell: \mathbb{R}^\nu \rightarrow \mathbb{C}$  be given as  $g(t) = |t - z|^{-2}$  and  $g_\ell(t) = t_\ell - \bar{z}_\ell$ . Write for  $2\beta \leq \alpha$

$$T_\alpha^\beta(t, z) := \frac{(-2)^{|\alpha-\beta|} |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (t - \text{Re } z)^{\alpha-2\beta} |t - z|^{-2|\alpha-\beta|}.$$

**Lemma 6.** Let  $g$  be as above and  $\alpha$  be any multi-index. Then

$$\partial^\alpha g(t) = \sum_{2\beta \leq \alpha} \alpha! T_\alpha^\beta(t, z) |t - z|^{-2}.$$

*Proof.* For brevity, we will write  $\alpha^i$  or  $\beta^i$  for  $\alpha + \delta_i$  or  $\beta + \delta_i$ , respectively. The formula is obviously true for  $|\alpha| \leq 1$ . Now assume that we have proven the formula for  $|\alpha| \leq k$ . Let  $|\alpha| = k$  and  $0 \leq i \leq \nu$  be arbitrary. It suffices to prove the formula for  $\alpha^i$ . One easily verifies using the chain rule that

$$(\partial^{\delta_i} g^n)(t) = -2n(t_i - \operatorname{Re} z_i)|t - z|^{-2n-2}. \quad (2)$$

Now by the induction hypothesis, we see that

$$\begin{aligned} \partial^{\alpha+\delta_i} g(t) &= \partial_t^{\delta_i} \sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha-\beta|} |\alpha|! |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (t - \operatorname{Re} z)^{\alpha-2\beta} |t - z|^{-2|\alpha-\beta|-2} \\ &= \sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha-\beta|} |\alpha|! |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (\partial_t^{\delta_i} (t - \operatorname{Re} z)^{\alpha-2\beta}) |t - z|^{-2|\alpha-\beta|-2} \end{aligned} \quad (3)$$

$$+ \sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha-\beta|} |\alpha|! |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (t - \operatorname{Re} z)^{\alpha-2\beta} (\partial_t^{\delta_i} |t - z|^{-2|\alpha-\beta|-2}). \quad (4)$$

For the sake of clarity, we will now consider each sum independently.

$$\begin{aligned} (3) &= \sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha-\beta|} |\alpha|! |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (\alpha_i - 2\beta_i) (t - \operatorname{Re} z)^{\alpha-2\beta-\delta_i} |t - z|^{-2|\alpha-\beta|-2} \\ &= \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} 2(\beta_i + 1) \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2} \\ &= \sum_{2\beta \leq \alpha+\delta_i} 2\beta_i \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2}. \end{aligned} \quad (5)$$

Using (2), we see that (4) equals

$$\begin{aligned} &\sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha-\beta|} |\alpha|! |\alpha-\beta|!}{2^{|\beta|} \beta! (\alpha-2\beta)!} (t - \operatorname{Re} z)^{\alpha-2\beta} (-2)(|\alpha - \beta| + 1)(t_i - \operatorname{Re} z_i) |t - z|^{-2|\alpha-\beta|-4} \\ &= \sum_{2\beta \leq \alpha} (\alpha_i + 1 - 2\beta_i) \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2} \\ &= \sum_{2\beta \leq \alpha} \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2} \end{aligned} \quad (6)$$

$$- \sum_{2\beta \leq \alpha} 2\beta_i \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2}. \quad (7)$$

Now (7) cancels (5) except for possible terms with  $2\beta = \alpha + \delta_i$ :

$$(5) + (7) = \sum_{2\beta = \alpha + \delta_i} \frac{(-2)^{|\alpha^i-\beta^i|} |\alpha^i|! |\alpha^i-\beta^i|!}{2^{|\beta^i|} \beta^i! (\alpha^i-2\beta^i)!} (t - \operatorname{Re} z)^{\alpha^i-2\beta^i} |t - z|^{-2|\alpha^i-\beta^i|-2}. \quad (8)$$

Adding (6) and (8) finishes the induction.  $\square$

**Lemma 7.** Let  $B \in C^{n_0}(A)$  for some  $n_0 \geq 1$  and let  $n \in \mathbb{N}_0$  and  $\alpha_0$  be a multi-index satisfying  $|\alpha_0| + n + 1 \leq n_0$ . Then

$$[\operatorname{ad}_A^{\alpha_0}(B), g(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha g(A) \operatorname{ad}_A^{\alpha_0+\alpha}(B) + R_n^g(A, \operatorname{ad}_A^{\alpha_0}(B)), \quad (9)$$

where

$$R_n^g(A, \text{ad}_A^{\alpha_0}(B)) = \sum_{\substack{|\alpha|=n-1 \\ 2\beta \leq \alpha}} \sum_{i=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z) \text{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) |A-z|^{-2} \quad (10)$$

$$+ \sum_{\substack{|\alpha|=n \\ 2\beta \leq \alpha}} \sum_{i=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z) (A_i - \bar{z}_i) \text{ad}_A^{\alpha_0+\alpha+\delta_i}(B) |A-z|^{-2} \quad (11)$$

$$+ \sum_{\substack{|\alpha|=n \\ 2\beta \leq \alpha}} \sum_{i=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z) \text{ad}_A^{\alpha_0+\alpha+\delta_i}(B) (A_i - z_i) |A-z|^{-2}. \quad (12)$$

*Proof.* The proof goes by induction. One may check by inspection of the following identity that the statement is true for  $n = 0$ .

$$\begin{aligned} [\text{ad}_A^{\alpha_0}(B), |A-z|^{-2}] &= - \sum_{i=1}^{\nu} |A-z|^{-2} (A_i - \bar{z}_i) \text{ad}_A^{\alpha_0+\delta_i}(B) |A-z|^{-2} \\ &\quad - \sum_{i=1}^{\nu} |A-z|^{-2} \text{ad}_A^{\alpha_0+\delta_i}(B) (A_i - z_i) |A-z|^{-2}. \end{aligned} \quad (13)$$

Now assume that we have proven the formula for  $k \leq n$ ,  $|\alpha_0| + n + 2 \leq n_0$ . We will now show that this implies that the formula holds for  $k = n + 1$ . We begin by noting two useful identities.

$$T_{\alpha}^{\beta}(t, z) |t-z|^{-2} = - \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(t, z). \quad (14)$$

$$(\beta_i + 1) T_{\alpha+2\delta_i}^{\beta+\delta_i}(t, z) 2(t_i - \text{Re } z_i) = (\alpha_i + 1 - 2\beta_i) T_{\alpha+\delta_i}^{\beta}(t, z). \quad (15)$$

Now using (13) and (14) we see that

$$(10) = \sum_{|\alpha|=n-1} \sum_{2\beta \leq \alpha} \sum_{i=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z) |A-z|^{-2} \text{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) \quad (16)$$

$$\begin{aligned} &+ \sum_{|\alpha|=n-1} \sum_{2\beta \leq \alpha} \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} \frac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A, z) \\ &\quad \times (A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B) |A-z|^{-2} \end{aligned} \quad (17)$$

$$\begin{aligned} &+ \sum_{|\alpha|=n-1} \sum_{2\beta \leq \alpha} \sum_{i=1}^{\nu} \sum_{j=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} \frac{\beta_j+\delta_{ij}+1}{|\alpha+\delta_i+\delta_j-\beta|} T_{\alpha+2\delta_i+2\delta_j}^{\beta+\delta_i+\delta_j}(A, z) \\ &\quad \times \text{ad}_A^{\alpha_0+\alpha+2\delta_i+\delta_j}(B) (A_j - z_j) |A-z|^{-2}, \end{aligned} \quad (18)$$

and by reordering and reindexing the sum in (16), (17) and (18), we get

$$(16) = \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 2}} \sum_{\substack{2\beta \leq \alpha \\ \beta_i \geq 1}} \frac{\beta_i}{|\alpha-\beta|} T_{\alpha}^{\beta}(A, z) |A-z|^{-2} \text{ad}_A^{\alpha_0+\alpha}(B), \quad (19)$$

and (17) equals

$$\sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 2}} \sum_{\substack{2\beta \leq \alpha \\ \beta_i \geq 1}} \sum_{j=1}^{\nu} \frac{\beta_i}{|\alpha-\beta|} \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A, z) (A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B) |A-z|^{-2} \quad (20)$$

and similarly for (18) with the factor  $(A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B)$  replaced by the factor  $\text{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j)$ . Note that we may relax the extra conditions on  $\alpha$  and  $\beta$  in the above statements, as a term with  $\beta_i = 0$  contributes nothing.

Instead of continuing in the same fashion with (11) and (12), we note using (15) that

$$(11) + (12) = \sum_{|\alpha|=n} \sum_{2\beta \leq \alpha} \sum_{i=1}^{\nu} \frac{\beta_i+1}{|\alpha+\delta_i-\beta|} T_{\alpha+2\delta_i}^{\beta+\delta_i}(A, z) \text{ad}_A^{\alpha_0+\alpha+2\delta_i}(B) |A-z|^{-2} \quad (21)$$

$$+ \sum_{|\alpha|=n} \sum_{2\beta \leq \alpha} \sum_{i=1}^{\nu} \frac{\alpha_i+1-2\beta_i}{|\alpha+\delta_i-\beta|} T_{\alpha+\delta_i}^{\beta}(A, z) \text{ad}_A^{\alpha_0+\alpha+\delta_i}(B) |A-z|^{-2}, \quad (22)$$

so we may focus our attention on (22):

$$(22) = \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \frac{\alpha_i-2\beta_i}{|\alpha-\beta|} T_{\alpha}^{\beta}(A, z) |A-z|^{-2} \text{ad}_A^{\alpha_0+\alpha}(B) \quad (23)$$

$$+ \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \frac{\alpha_i-2\beta_i}{|\alpha-\beta|} \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A, z) \times (A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B) |A-z|^{-2}. \quad (24)$$

$$+ \sum_{i=1}^{\nu} \sum_{\substack{|\alpha|=n+1 \\ \alpha_i \geq 1}} \sum_{\substack{2\beta \leq \alpha \\ 2\beta_i < \alpha_i}} \sum_{j=1}^{\nu} \frac{\alpha_i-2\beta_i}{|\alpha-\beta|} \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A, z) \times \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j) |A-z|^{-2} \quad (25)$$

We note again that the additional conditions on  $\alpha$  and  $\beta$  are superfluous.

We may now recollect the terms. First we see using Lemma 6:

$$\sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^{\alpha} g(A) \text{ad}_A^{\alpha_0+\alpha}(B) + (19) + (23) = \sum_{|\alpha|=1}^{n+1} \frac{1}{\alpha!} \partial^{\alpha} g(A) \text{ad}_A^{\alpha_0+\alpha}(B), \quad (26)$$

then

$$(20) + (24) = \sum_{\substack{|\alpha|=n+1 \\ 2\beta \leq \alpha}} \sum_{j=1}^{\nu} \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A, z) (A_j - \bar{z}_j) \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B) |A-z|^{-2}, \quad (27)$$

and

$$(18) + (25) = \sum_{\substack{|\alpha|=n+1 \\ 2\beta \leq \alpha}} \sum_{j=1}^{\nu} \frac{\beta_j+1}{|\alpha+\delta_j-\beta|} T_{\alpha+2\delta_j}^{\beta+\delta_j}(A, z) \text{ad}_A^{\alpha_0+\alpha+\delta_j}(B)(A_j - z_j) |A-z|^{-2}, \quad (28)$$

so adding up, we have proved that (9) equals the sum of (26), (21), (27) and (28) as stated.  $\square$

The following lemma plays the same role for  $g_\ell$  as Lemma 7 plays for  $g$ , but contrary to Lemma 7, the proof is trivial.

**Lemma 8.** *Let  $B \in C^{n_0}(A)$  for some  $n_0 \geq 1$  and let  $n \in \mathbb{N}_0$  and  $\alpha_0$  be a multi-index satisfying  $|\alpha_0| + n + 1 \leq n_0$ . Then*

$$[\text{ad}_A^{\alpha_0}(B), g_\ell(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha g_\ell(A) \text{ad}_A^{\alpha_0+\alpha}(B) + R_n^{g_\ell}(A, \text{ad}_A^{\alpha_0}(B)),$$

where  $R_n^{g_\ell}(A, \text{ad}_A^{\alpha_0}(B)) = 0$  for  $n \geq 1$ ,  $R_0^{g_\ell}(A, \text{ad}_A^{\alpha_0}(B)) = \text{ad}_A^{\alpha_0+\delta_\ell}(B)$ .

The following lemma also follows by induction.

**Lemma 9.** *Let  $B \in C^{n_0}(A)$  for some  $n_0 \geq 1$ . Assume that  $h_i \in C^\infty(\mathbb{R}^\nu)$ ,  $1 \leq i \leq k$ , satisfies*

$$[\text{ad}_A^{\alpha_0}(B), h_i(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha h_i(A) \text{ad}_A^{\alpha_0+\alpha}(B) + R_n^{h_i}(A, \text{ad}_A^{\alpha_0}(B)),$$

where  $R_n^{h_i}(A, \text{ad}_A^{\alpha_0}(B))$  is bounded for all  $n \in \mathbb{N}_0$  and multi-indices  $\alpha_0$  satisfying  $|\alpha_0| + n + 1 \leq n_0$  and  $\partial^\alpha h_i(A)$  is bounded for all  $1 \leq |\alpha| \leq n_0 - 1$ . Then

$$\begin{aligned} \left[ B, \prod_{i=1}^k h_i(A) \right] &= \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha \left( \prod_{i=1}^k h_i \right)(A) \text{ad}_A^\alpha(B) \\ &\quad + \sum_{j=1}^k \sum_{|\alpha|=0}^n \frac{1}{\alpha!} \partial^\alpha \left( \prod_{i=1}^{j-1} h_i \right)(A) R_{n-|\alpha|}^{h_j}(A, \text{ad}_A^\alpha(B)) \prod_{i=j+1}^k h_i(A). \end{aligned}$$

Let  $n + 1 \leq n_0$ . If we put  $k = \nu + 1$ ,  $h_i = g$  for  $i \neq \nu$ ,  $h_\nu = g_\ell$  and apply Lemma 7, 8 and 9 we see that

$$\begin{aligned} [B, |A - z|^{-2\nu}(A_\ell - \bar{z}_\ell)] &= \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha (|\cdot - z|^{-2\nu}(\cdot_\ell - \bar{z}_\ell))(A) \text{ad}_A^\alpha(B) + R_{\ell,n}(A, B), \end{aligned} \quad (29)$$

where

$$R_{\ell,n}(A, B) = \sum_{j=1}^{\nu-1} \sum_{|\alpha|=0}^n \frac{1}{\alpha!} \partial^\alpha (g^{j-1})(A) R_{n-|\alpha|}^g(A, \text{ad}_A^\alpha(B)) |A - z|^{-2(\nu-j)} (A_\ell - \bar{z}_\ell) \quad (30)$$

$$+ \sum_{|\alpha|=n} \frac{1}{\alpha!} \partial^\alpha (g^{\nu-1})(A) \text{ad}_A^{\alpha+\delta_\ell}(B) |A - z|^{-2} \quad (31)$$

$$+ \sum_{|\alpha|=0}^n \frac{1}{\alpha!} \partial^\alpha (g^{\nu-1} g_\ell)(A) R_{n-|\alpha|}^g(A, \text{ad}_A^\alpha(B)) \quad (32)$$

In the following, we will refer to the terms of  $R_{\ell,n}(A, B)$  as the remainder terms. Let  $0 \leq t_1 \leq n+1$  and  $0 \leq t_2 \leq 1$ . By Hadamard's three-line lemma and using (10–12), (30–32), Lemma 6 and the identity

$$\partial^\alpha \left( \prod_{i=1}^j f_i \right) = \sum_{\sum \alpha_i = \alpha} \frac{\alpha!}{\prod_{i=1}^j \alpha_i!} \prod_{i=1}^j \partial^{\alpha_i} f_i,$$

we may inspect that each remainder term (with  $R_{\ell,n}(A, B)$  replaced by the remainder term) and hence  $R_{\ell,n}(A, B)$  satisfies the inequality

$$\| \langle A \rangle^{t_1} R_{\ell,n}(A, B) \langle A \rangle^{t_2} \| \leq C \langle z \rangle^{t_1+t_2} |\operatorname{Im} z|^{-n-2\nu}. \quad (33)$$

We will now use the functional calculus of almost analytic extensions. See e.g. [3] for details. In the following, we write  $\bar{\partial} = (\bar{\partial}_1, \dots, \bar{\partial}_\nu)$  where  $\bar{\partial}_j = \frac{1}{2}(\partial_{u_j} + i\partial_{v_j})$  and  $u_j + v_j = z_j \in \mathbb{C}$ ,  $z = (z_1, \dots, z_n) \in \mathbb{C}^\nu$ . The following proposition is inspired by [4] and [8, Chap. X.2].

**Proposition 10.** *Let  $s \in \mathbb{R}$  and  $\{f_\lambda\}_{\lambda \in I} \subset C^\infty(\mathbb{R}^\nu)$  satisfy*

$$\forall \alpha \exists C_\alpha: |\partial^\alpha f_\lambda(x)| \leq C_\alpha \langle x \rangle^{s-|\alpha|}.$$

*There exists a family of almost analytic extensions  $\{\tilde{f}_\lambda\}_{\lambda \in I} \subset C^\infty(\mathbb{C}^\nu)$  satisfying*

$$(i) \operatorname{supp}(\tilde{f}_\lambda) \subset \{u + iv \mid u \in \operatorname{supp}(f_\lambda), |v| \leq C \langle u \rangle\}.$$

$$(ii) \forall \ell \geq 0 \exists C_\ell: |\bar{\partial}^\ell \tilde{f}_\lambda(z)| \leq C_\ell \langle z \rangle^{s-\ell-1} |\operatorname{Im} z|^\ell.$$

*Proof.* We define a mapping  $C^\infty(\mathbb{R}^\nu) \ni f \mapsto \tilde{f} \in C^\infty(\mathbb{C}^\nu)$  in the following way. Choose a function  $\kappa \in C_0^\infty(\mathbb{R})$  which equals 1 in a neighbourhood of 0 and put  $\lambda_0 = C_0$ ,  $\lambda_k = \max\{\max_{|\alpha|=k} C_\alpha, \lambda_{k-1} + 1\}$  for  $k \geq 1$ . Writing  $z = u + iv \in \mathbb{R}^\nu \oplus i\mathbb{R}^\nu$ , we now define

$$\tilde{f}(z) = \sum_{\alpha} \frac{\partial^\alpha f(u)}{\alpha!} (iu)^\alpha \prod_{j=1}^{\nu} \kappa\left(\frac{\lambda_{|\alpha|} v_j}{\langle u \rangle}\right).$$

One can now check that the properties hold.  $\square$

**Remark 11.** Note that if we for a  $\chi \in C_0^\infty(\mathbb{R}^\nu; [0, 1])$  with  $\chi(0) = 1$  define a sequence of functions by  $f_{k,\lambda}(x) = \chi(\frac{x}{k}) f_\lambda(x)$ , then

$$[B, f_\lambda(A)] = \lim_{k \rightarrow \infty} [B, f_{k,\lambda}(A)]$$

as a form identity on  $\mathcal{D}(\langle A \rangle^s)$  and we have the dominated pointwise convergence

$$\bar{\partial} \tilde{f}_{k,\lambda}(x) \rightarrow \bar{\partial} \tilde{f}_\lambda(x) \text{ for } k \rightarrow \infty.$$

Let  $\{f_\lambda\}_{\lambda \in I}$  satisfy the assumption of Proposition 10 with  $s < 0$ . Then the almost analytic extensions provide a functional calculus via the formula

$$f_\lambda(A) = C_\nu \sum_{\ell=1}^{\nu} \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) (A_\ell - \bar{z}_\ell) |A - z|^{-2\nu} dz, \quad (34)$$



where  $C_\nu$  is a positive constant (again we refer to [3] for details). Note that the integrals are absolutely convergent by Proposition 10(ii).

Multiplying  $\langle A \rangle^{t_1} R_{\ell,n}(A, B) \langle A \rangle^{t_2}$  with  $\bar{\partial} \tilde{f}_\lambda(z)$ , we get from (33) and Proposition 10 (ii) that

$$\|\langle A \rangle^{t_1} \bar{\partial} \tilde{f}_\lambda(z) R_{\ell,n}(A, B) \langle A \rangle^{t_2}\| \leq C \langle z \rangle^{t_1+t_2+s-n-1-2\nu}. \quad (35)$$

Hence, if  $t_1 + t_2 + s < n + 1$ ,  $\langle A \rangle^{t_1} \bar{\partial} \tilde{f}_\lambda(z) R_{\ell,n}(A, B) \langle A \rangle^{t_2}$  is integrable over  $\mathbb{C}^\nu$ . Using (29), (34) and (35), we see that

$$\begin{aligned} [B, f_\lambda(A)] &= C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) [B, (A_\ell - \bar{z}_\ell) |A - z|^{-2\nu}] dz \\ &= C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha (|\cdot - z|^{-2\nu} (\cdot_\ell - \bar{z}_\ell))(A) dz \operatorname{ad}_A^\alpha(B) \\ &\quad + C_\nu \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) R_{\ell,n}(A, B) dz. \end{aligned} \quad (36)$$

We denote (36) by  $R_{\lambda,n}(A, B)$ . Note that

$$\begin{aligned} &\sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) \frac{1}{\alpha!} \partial_t^\alpha (|t - z|^{-2\nu} (t_\ell - \bar{z}_\ell)) dz \\ &= \frac{1}{\alpha!} \partial_t^\alpha \sum_{\ell=1}^\nu \int_{\mathbb{C}^\nu} \bar{\partial}_\ell \tilde{f}_\lambda(z) |t - z|^{-2\nu} (t_\ell - \bar{z}_\ell) dz = \frac{1}{\alpha!} \partial^\alpha f_\lambda(t), \end{aligned}$$

which implies

$$[B, f_\lambda(A)] = \sum_{|\alpha|=1}^n \frac{1}{\alpha!} \partial^\alpha f_\lambda(A) \operatorname{ad}_A^\alpha(B) + R_{\lambda,n}(A, B).$$

We have now proved Theorem 3 in the case  $s < 0$ . For the general case, we use Remark 11 to see that  $[B, f_\lambda(A)] = \lim_{k \rightarrow \infty} [B, f_{k,\lambda}(A)]$  and clearly,  $f_{k,\lambda}$  satisfies the assumption of Proposition 10 with the same  $s$ , so the estimate corresponding to (35) is now uniform in  $k$  and  $\lambda$ . The pointwise convergence and Lebesgue's theorem on dominated convergence now finishes the argument.

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